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ADDENDUM

More on the state-dependent implication in quantum mechanics

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Abstract. The mixing structure of a given quantum state (statistical operator) ρ is considered, that is, the pure states and associated weights of which it consists. Its influence on the state-dependent quantum logical implication determined by ρ is also studied. It is found that it has no influence at all except through the null-projector. It is shown that the latter, when it determines the state-dependent implication, in a certain sense, carries with it all the Boolean mathematical structure of projectors in the given Boolean subalgebra $\mathcal{B}$ of quantum logic $\mathcal{P}(\mathcal{H})$, where $\mathcal{B}$ is the 'domain' of definition of the state-dependent implication.

A general (mixed or pure) quantum-mechanical state (statistical operator) ρ is usually decomposed into pure states (though not uniquely)

$$\rho = \sum_i w_i |\psi_i\rangle\langle\psi_i| \qquad \forall i : w_i > 0, \sum_i w_i = 1 \tag{1}$$

where $\{|\psi_i\rangle : i = 1, 2, \ldots\}$ are state vectors and the sum is finite or infinite. A special case is a spectral form of ρ, when w_i are the positive characteristic values and $|\psi_i\rangle$ are corresponding characteristic vectors.

We investigate in this note to what extent the implication '$\rightarrow_\rho$' determined by ρ (cf Herbut 1994) is dependent on the mixing structure (1) of ρ.

Let us start by outlining an elementary, but not so well known fact. Any preorder '$\rightarrow$' (a binary relation that is reflexive and transitive, see Birkhoff (1940)) in a set S induces an equivalence relation '$\sim\!\rightarrow$' (a binary relation that is reflexive, symmetric and transitive):

$$E, F \in S : E \sim\!\rightarrow F \qquad \text{if both } E \rightarrow F \text{ and } F \rightarrow E.$$

Further, the preorder induces an order '$\rightarrow$' (binary relation that is reflexive, transitive and antisymmetric) on the quotient set $S/\sim\!\rightarrow$, consisting of the equivalence classes, via an arbitrary representative:

$$E, F \in S \qquad [E], [F] \in (S/\sim\!\rightarrow) : [E] \rightarrow [F] \qquad \text{if } E \rightarrow F$$

where $[E]$ is the equivalence class to which E belongs etc. Thus, every preorder '$\rightarrow$' in S decomposes into an equivalence relation '$\sim$' in S and into an order '$\rightarrow$' in $S/\sim$ and, *vice versa*, any of the latter two, given as described, defines a former relation.

As was stated in a recent article (definition 3 in Herbut (1994)), an *implication* (a notion meant to extend the absolute implication) in a given Boolean subalgebra $\mathcal{B}$ of quantum logic $\mathcal{P}(\mathcal{H})$ is a preorder '$\rightarrow$' such that the equivalence relation '$\sim\!\rightarrow$' induced by it makes the

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equivalence class [0] of the zero projector $P \equiv 0$ an ideal $\Delta(= [0])$, furthermore, such that '$\sim\rightarrow$'='$\sim_\Delta$' (cf (2) in Herbut (1994)), and finally, such that the order '$\rightarrow$' induced in the partially ordered quotient set $\mathcal{B}/\sim\rightarrow$ amounts to the same as the absolute implication in the factor algebra $\mathcal{B}/\Delta(= \mathcal{B}/\sim\rightarrow)$.

Thus, *all implications* in $\mathcal{B}$ are actually *determined by the ideals* in $\mathcal{B}$. The state-dependent implication '$\rightarrow_\rho$' is determined by the ideal in $\mathcal{B}$ defined by ρ:

$$\Delta_\rho \equiv \{E : E \in \mathcal{B}, E \leqslant Q_0\} \tag{2}$$

where Q_0 is the null-projector of ρ (cf theorem 1 and section 2 in Herbut (1994)).

Thus, *no detail* of the mixing structure (1) of ρ has any influence on '$\rightarrow_\rho$' except the null-projector

$$Q_0 = 1 - \sum_i |\psi_i\rangle\langle\psi_i| \tag{3}$$

where now $\{|\psi_i\rangle : i = 1, 2, \ldots\}$ are orthonormal characteristic vectors of ρ spanning its range. More precisely, one has the following result.

Theorem 1. Two statistical operators ρ and ρ' determine one and the same state-dependent implication in a given Boolean subalgebra $\mathcal{B}$ of quantum logic $\mathcal{P}(\mathcal{H})$, i.e., '$\leqslant_\rho$'='$\leqslant_{\rho'}$', if and only if ρ and ρ' have one and the same null-projector, i.e., $Q_0 = Q_0'$.

Proof. Sufficiency. If $Q_0 = Q_0'$, then $\Delta_\rho = \Delta_\rho$, (cf (2)), and $\mathcal{B}/\Delta_\rho = \mathcal{B}/\Delta_{\rho'}$. Then necessarily '$\leqslant_\rho$'='$\leqslant_{\rho'}$' because this Boolean factor algebra and the order induced in it by the absolute implication '$\leqslant$' in $\mathcal{B}$ determines the implication '$\leqslant_\rho$' (and '$\leqslant_{\rho'}$').

Necessity. If '$\leqslant_\rho$'='$\leqslant_{\rho'}$', then '$\mathcal{B}/\sim_\rho$'='$\mathcal{B}/\sim_{\rho'}$' or, equivalently, '$\mathcal{B}/\Delta_\rho$'='$\mathcal{B}/\Delta_{\rho'}$', in particular, $\Delta_\rho = \Delta_{\rho'}$, implying, in view of (2), $Q_0 = Q_0'$. □

Remark 1. The state-dependent implication '$\rightarrow_\rho$' determined by a given statistical operator ρ in a Boolean subalgebra $\mathcal{B}$ of $\mathcal{P}(\mathcal{H})$, as well as the corresponding equivalence relation '$\sim_\rho$', can be expressed in terms of the null-projector Q_0 (and the structure of $\mathcal{B}$):

$$E, F \in \mathcal{B}, E \sim_\rho F \qquad \text{if and only if } E^\perp F, EF^\perp \leqslant Q_0$$

(cf (2) and the definition of a Boolean factor algebra $\mathcal{B}/\Delta$, see Herbut (1994), relation (2)). Finally,

$$E, F \in \mathcal{B}, E \leqslant_\rho F \qquad \text{if and only if } \exists E_0 : E_0 \sim_\rho E, \text{ and } \exists F_0 : F_0 \sim_\rho F, \text{ and } E_0 \leqslant F_0.$$

Actually, every projector $E(\in \mathcal{B})$, except $E \equiv 1$, is the null-projector of some statistical operator ρ, and it defines an ideal

$$\Delta_E \equiv \{F : F \in \mathcal{B}, F \leqslant E\}. \tag{4}$$

Remark 2. Let us denote by $\mathcal{I}(\mathcal{B})$ the set of all implications in $\mathcal{B}$. The binary relation '$\leqslant_i$' in $\mathcal{I}(\mathcal{B})$ defined by '$\rightarrow$' $\leqslant_i$'$\rightarrow$', '$\rightarrow$', '$\rightarrow$' $\in \mathcal{I}(\mathcal{B})$: if, whenever $E \rightarrow F$, $E, F \in \mathcal{B}$, then also $E \rightarrow' F$ is an order in $\mathcal{I}(\mathcal{B})$, making it a partially ordered set. We denote it by $(\mathcal{I}(\mathcal{B}), \leqslant_i)$.

Theorem 2. Each $E \in \mathcal{B}$ defines an implication '$\rightarrow_E$'$\in \mathcal{I}(\mathcal{B})$ (via the Boolean factor algebra $\mathcal{B}/\Delta_E$, cf (4)) acting as follows:

$$G \rightarrow_E H \qquad G, H \in \mathcal{B}, \qquad \text{if } [G]_E \leqslant [H]_E$$

where $[G]_E$, $[H]_E \in (\mathcal{B}/\Delta_E)$, and '$\leqslant$' is the absolute implication induced in $\mathcal{B}/\Delta_E$ by the absolute implication '$\leqslant$' in $\mathcal{B}$. Thus $\mathcal{B}$ is mapped into $\mathcal{I}(\mathcal{B})$. This map is an injection that, together with its inverse, is an isomorphism of the corresponding partially ordered sets. Thus, the partially ordered set $(\mathcal{B}, \leqslant)$ is *embedded* into $(\mathcal{I}(\mathcal{B}), \leqslant_i)$.

Proof. Let $E \neq F$, $E, F \in \mathcal{B}$. Since $EF^{\perp} = 0 \Leftrightarrow E \leqslant F$ (and symmetrically, $E^{\perp}F = 0 \Leftrightarrow F \leqslant E$), at least one of the two projectors $EF^{\perp}$ and $E^{\perp}F$ is non-zero. Let $EF^{\perp} \neq 0$ (otherwise the argument is symmetric to the one that follows). Nevertheless, $EF^{\perp} \in \Delta_E$ (cf (4)), but $EF^{\perp} \notin \Delta_F$. Hence, though $EF^{\perp} \rightarrow_E 0$, the relation $EF^{\perp} \rightarrow_F 0$ is not true. Thus, '$\rightarrow_E$'$\neq$'$\rightarrow_F$', and the map in question is one-to-one, i.e., it is an injection.

Let $E \leqslant F$, $E, F \in \mathcal{B}$, and let $G \rightarrow_E H$, $G, H \in \mathcal{B}$. Then $\exists G_0, H_0 \in \mathcal{B}$, such that $[G_0]_E = [G]_E$, $[H_0]_E = [H]_E$ and $[G]_E \leqslant [H]_E$. But since $\Delta_E \subseteq \Delta_F$ (cf (4)), one has $[G]_E \subseteq [G]_F$ and $[H]_E \subseteq [H]_F$. Hence, also $[G]_F \leqslant [H]_F$ and $G \rightarrow_F H$, implying '$\rightarrow_E$'$\leqslant_i$'$\rightarrow_F$'. Thus, the injection at issue preserves the partial order from $(\mathcal{B}, \leqslant)$ to $(\mathcal{I}(\mathcal{B}), \leqslant_i)$.

Finally, let '$\rightarrow_E$'$\leqslant_i$'$\rightarrow_F$'. Then whenever $G \rightarrow_E H$, $G, H \in \mathcal{B}$, also $G \rightarrow_F H$. Hence, also whenever $G \sim\!\rightarrow_E H$, then also $G \sim\!\rightarrow_E H$. Since $E \sim\!\rightarrow_E 0$, also $E \sim\!\rightarrow_F 0$. But then $E \leqslant F$ (cf (4)). Thus, also the inverse of the map in question is an isomorphism. □

The partially ordered set $(\mathcal{B}, \leqslant)$ has a remarkable structure in which the order '$\leqslant$'—the absolute implication —determines a Boolean algebra. It is, thus, faithfully transferred into the partially ordered set of implications $(\mathcal{I}(\mathcal{B}), \leqslant_i)$.

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